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STEADY-STATE AND PERIODIC MOTIONS IN THE ATTRACTION FIELD OF A ROTATING TRIAXIAL ELLIPSOID*

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The motion of a material point in the field of attraction of a rotating triaxial ellipsoid is examined in the presence of exact commensurability between the triaxial ellipsoid's angular rotation velocity and the average motion of the material point over the orbit. Families of periodic solutions of Schwarzschild type are found for certain commensurabilities of lower order and for values of a parameter characterizing the form of the ellipsoid. It is well known /1/ that the equations of motion of a mass point in the attraction field of a rotating triaxial ellipsoid have periodic solutions under different commensurabilities between the average motion of the mass point and the angular rotation velocity of the ellipsoid.In particular, to a commensurability of the kind 1/1 correspond steady-state motions (libration points) which exist also for bodies of a form more complex than a triaxial ellipsoid /2—5/. The conditions for the existence of the periodic solutions (orbits)mentioned, in the triaxial ellipsoid's attraction field have been investigated only for certain values of eccentricity and inclination. In this regard there is great interest in a more detailed investigation of existence conditions for steady-state, conditionally-periodic and periodic solutions of this problem when the eccentricity and the inclination vary within the limits $0<e<1$ and $0<i<\pi$, respectively.

1. Statement of the problem. Let a mass point move in the attraction field of a homogeneous (or inhomogeneous, but with ellipsoidal layers of identical density) triaxial ellipsoid (a planet) of mass M', rotating with constant angular velocity ω_0 around one of the principal central axes of inertia. We use the following coordinate systems: 1) a rectangular inertial system $Oxyz$ with origin at the ellipsoid's center of mass, where the axis Ox is directed into the vernal equinox, the plane Oxy coincides with the ellipsoid's equatorial plane, and the axis Oz is directed towards the world's north pole; 2) a rectangular rotating system $OXYZ$ with origin at the ellipsoid's center of mass, where the axis OX passes through some zero meridian and is located, together with axis OY, in the ellipsoid's equatorial plane and the axis OZ is directed so that is coincides with the fixed axis Oz and is at the same time the ellipsoid's rotation axis; 3) a spherical rotating system $r\varphi\lambda$ with radius-vector r, longitude λ read off to the east of the zero meridian, and lattitude φ read off to the north of the ellipsoid's equatorial plane.

The equations of motion of the mass point in the triaxial ellipsoid's attraction field, in a spherical coordinate system rotating with constant angular velocity ω_0, are of the form

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} - r\lambda_*^{\cdot 2}\cos^2\varphi = \partial V / \partial r \tag{1.1}$$

$$d(r^2\varphi^{\cdot})/dt + r^2\lambda_*^{\cdot}\sin\varphi\cos\varphi = \partial V/\partial\varphi, \qquad d(r^2\lambda_*^{\cdot}\cos^2\varphi)/dt = \partial V/\partial\lambda, \quad \lambda_*^{\cdot} = \lambda^{\cdot} + \omega_0$$

Here V is the triaxial ellipsoid's gravitational potential, written as

$$V = \frac{fM}{r}\left[1 + C_{20}\left(\frac{r_0}{r}\right)^2 P_2(\sin\varphi) + C_{22}^{\circ}\left(\frac{r_0}{r}\right)^2 P_{22}(\sin\varphi)\cos 2(\lambda - \lambda_{22})\right], \quad C_{22}^{\circ} = \sqrt{C_{22}^2 + S_{22}^2} \tag{1.2}$$

Here f is the gravitational constant, r_0 is the ellipsoid's largest equatorial radius, C_{20}, C_{22} and S_{22} are coefficients characterizing the form of the triaxial ellipsoid, λ_{22} is the angle characterizing the orientation of the major semiaxis of the equatorial section of the ellipsoid relative to the zero meridian, $P_2(\sin\varphi)$ and $P_{22}(\sin\varphi)$ are the Legendre polynomial and the Legendre associated function, respectively. The expansion of potential V in the form (1.2)is due to the choice of the coordinate systems $r\varphi\lambda$ and $OXYZ$, and is obtained under the assumption that axis OZ coincides with a principal central inertia axis of the ellipsoid, with due regard to triaxial ellipsoid's second-order moments of inertia. The problem is to find steady-state and periodic solutions of equation system (1.1) defining the motion of a mass point in the attraction field of a triaxial ellipsoid with gravitational potential (1.2).

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2. Passage to canonic variables. We select one of the Keplerian systems of elements, for example, $p\,(a)$, e, i, Ω, ω and v, where $p\,(a)$ is a parameter (major semiaxis) of the orbit, e is the eccentricity, i is the inclination, Ω is the longitude of the orbit's ascension angle, read off from the fixed axis Ox, ω is the argument of the perigee, and v is the true anomaly, and we pass, in accord with /4/, to dimensionless variables by the formulas

$$p^* = pp_0^{-1},\quad r^* = rp_0^{-1},\quad \tau = \mu^{1/2}p_0^{-3/2}t,\qquad n^* = \mu^{-1/2}p_0^{3/2}n,\quad \omega_0^* = \mu^{-1/2}p_0^{3/2}\omega_0,\quad \mu = fM'$$

so that $\omega_0^* = p_0^* = \mu = 1$. Here n is the average motion, p_0 is the orbit's parameter at the initial instant, t is time.

We introduce a small parameter $\varkappa$ by the formula

$$\varkappa = -3C_{20}\,(r_0\,/\,p_0)^2$$

We write the problem's perturbing function as

$$U = \frac{p_0}{\mu\varkappa}\left(V - \frac{\mu}{r}\right)$$

and we use the expansions /6/

$$\left(\frac{r}{a}\right)^n \cos mv = \sum_{k=0} C_k^{n,\,m}\cos kM,\qquad \left(\frac{r}{a}\right)^n \sin mv = \sum_{k=0} S_k^{n,\,m}\sin kM$$

($C_k^{n,m}$ and $S_k^{n,m}$ are power series in the eccentricity e, M is the mean anomaly). After the transformations indicated have been effected the equation system (1.1) takes the Lagrange form, while the perturbing function U is written as

$$U = \sum_{\|\mathbf{k}\|=0} a_k(a^*, e, i)\cos(k_1M + k_2\Omega + k_3\omega + k_4\tau),\qquad \mathbf{k} = (k_1, k_2, k_3, k_4),\quad \|\mathbf{k}\| = \sum_{\nu=1}^{4}|k_\nu|$$

$$a_k = a^{*-3}f_k,\quad f_{k_1000} = {}^1/_6\,(1 - {}^3/_2\sin^2 i)\,C_{k_1}^{-3,0},\qquad f_{k_10,\pm2,0} = {}^1/_8\sin^2 i\,(C_{k_1}^{-3,2} \pm S_{k_1}^{-3,2})$$

$$k_2 \neq 0,\quad f_k = {}^3/_8\varkappa b_{22}g_k,\quad b_{22} = C_{22}^{\circ}\varkappa^{-2}\,(r_0\,/\,p_0)^2,\quad g_{k_1,10,-1} = 2\sin^2 i\,C_{k_1}^{-3,0},\quad g_{k_1,-1,01} = g_{k_110,-1}$$

$$g_{k_1,\pm2,\pm2,\mp2} = (1 + \cos i)^2(C_{k_1}^{-3,2} \pm S_{k_1}^{-3,2}),\qquad g_{k_1,\pm2,\mp2,\mp2} = (1 - \cos i)^2(C_{k_1}^{-3,2} \pm S_{k_1}^{-3,2})$$

The structure of the perturbing function is such that $k_3 = -k_4$. Therefore, by introducing Delaunay's canonic elements

$$x_1 = L = \sqrt{a^*},\quad y_1 = l = M,\qquad x_2 = G = \sqrt{a^*\,(1 - e^2)},\quad y_2 = g = \omega \tag{2.1}$$

$$x_3 = H = \sqrt{a^*\,(1 - e^2)}\cos i,\quad y_3 = h - \tau = \Omega - \tau$$

and analyzing the motion in the rotating coordinate system $OXYZ$, we write the equations of motion of the mass point in the autonomous canonic form

$$dx_j\,/\,d\tau = \partial F'\,/\,\partial y_j,\quad dy_j\,/\,d\tau = -\partial F'\,/\,\partial x_j\quad (j = 1, 2, 3) \tag{2.2}$$

$$F' = F_0' + \varkappa F_1',\quad F_0' = {}^1/_2 x_1^{-2} + x_3,\qquad -F_1' = U = \sum_{\|\mathbf{k}\|\geqslant 0} A_k(x_j)\cos(k_1y_1 + k_2y_2 + k_3y_3)$$

Here F' is the system's Hamiltonian; the coefficients $A_k(x_j)$ are determined from the coefficients $a_k(a^*, e, i)$, allowing for (2.1).

We shall examine the resonance cases of motion of the mass point, when a sharp commensurability occurs between the average motion of the mass point on the orbit and the triaxial ellipsoid's angular rotation velocity. The commensurability condition can be written as $n^* = (p + q)\,/\,p$, where p and q are integers. To investigate the steady-state and periodic solutions in the resonance cases we introduce the following system of canonic variables:

$$X_1 = sL - pH,\quad X_2 = G - H,\quad X_3 = H,\qquad Y_1 = l\,/\,s,\quad Y_2 = g,\quad Y_3 = pl\,/\,s + g + h - \tau$$

Here Y_3 is the critical argument or the Delaunay anomaly; $s = p + q$. The differential equations (2.2) in the variables X_j and Y_j also have a canonic form

$$dX_j\,/\,d\tau = \partial F\,/\,\partial Y_j,\quad dY_j\,/\,d\tau = -\partial F\,/\,\partial X_j\quad (j = 1, 2, 3),\quad F = F_0 + \varkappa F_1,\quad F_0 = {}^1/_2 s^2\,(X_1 + pX_3)^{-2} + X_3 \tag{2.3}$$

$$-F_1 = \sum_{\|k\|\geqslant 0} A_k^*(X_j)\cos[(k_1s + mp)Y_1 + k_2Y_2 + k_3Y_3],\qquad m = 0, \pm1, \pm2$$

3. Steady-state solutions. We seek the steady-state solutions of equation system (2.3). Using the Tseipel transformation we eliminate from the perturbing part F_1 of Hamiltonian F the short-period terms, i.e., the terms containing the fast variable Y_1, and we write F_1 as

$$F_1^* = F_1^{\text{sec}} + F_1^{\text{res}},\qquad F_1^{\text{sec}} = \frac{s^6}{12\,(X_1 + pX_3)^6}(3\alpha - 1)\,C_0^{-3,2}(X_j),\qquad F_1^{\text{res}} = \frac{3\varkappa b_{22}s^6}{4\,(X_1 + pX_3)^6}\sum_{i=1}^{3}\Phi_i$$

$$\Phi_1 = (1 - \alpha^2)C_k^{-3,0}(X_j)\cos[(ks - 2p)Y_1 - 2Y_2 + 2Y_3]$$

$$\Phi_2 = {}^1/_2\,(1 + \alpha)^2[C_k^{-3,2}(X_j) + S_k^{-3,2}(X_j)]\cos[(ks - 2p)Y_1 + 2Y_3]$$

$$\Phi_3 = {}^1/_2\,(1 - \alpha)^2[C_k^{-3,2}(X_j) - S_k^{-3,2}(X_j)]\cos[(ks - 2p)Y_1 - 4Y_2 + 2Y_3], \quad \alpha = X_3/(X_2 + X_3)$$

Thus, F_1^{sec} has one and the same form under all commensurabilities, while F_1^{res} depends upon the form of the commensurability, in particular, on the values of k, s and p. Table 1 shows the number of the harmonics occurring in the corresponding F_1^{res} for the commensurabilities most important for the given problem.

Table 1

p	q	$(p+q)/p$	k
1	0	1/1	2
2	−1	1/2	4
3	−2	1/3	6
4	−3	1/4	8
5	−4	1/5	10
1	1	2/1	1
3	−1	2/3	3

We shall seek the steady-state solutions $X_j = \text{const}$ $(j = 1, 2, 3)$ and $Y_i = \text{const}$ $(i = 2, 3)$ of equation system (2.3) with Hamiltonian $F^* = F_0 + \varkappa F_1^*$. In this case Eqs. (2.3) are written as

$$\frac{dX_1}{d\tau} = 0, \quad \frac{dY_1}{d\tau} = s^{-1}\left(n^* - \varkappa\frac{\partial F_1^*}{\partial L}\right), \quad \frac{dX_2}{d\tau} = \varkappa\frac{\partial F_1^{res}}{\partial Y_2} = 0, \quad \frac{dY_2}{d\tau} = -\varkappa\frac{\partial F_1^*}{\partial G} = 0 \qquad (3.1)$$

$$\frac{dX_3}{d\tau} = \varkappa\frac{\partial F_1^{res}}{\partial Y_3} = 0, \quad \frac{dY_3}{d\tau} = \left\{n^* - \frac{s}{p} - \varkappa\left[\frac{\partial F_1^*}{\partial L} + \frac{s}{p}\left(\frac{\partial F_1^*}{\partial G} + \frac{\partial F_1^*}{\partial H}\right)\right]\right\} = 0$$

Bear in mind here that

$$s^3(X_1 + pX_3)^{-3} = L^{-3} = n^*$$

The structure of function F_1^{res} for the commensurabilities considered (see Table 1) are such that the third and the fifth equations in (3.1) are satisfied for the values of the angular variables Y_2 and Y_3 shown in Table 2, and, thus, there are 16 types of solutions.

Table 2

Type of solution	Y_2	Y_3	Type of solution	Y_2	Y_3	Type of solution	Y_2	Y_3	Type of solution	Y_2	Y_3
1	0	0	5	0	$\pi/2$	9	0	π	13	0	$3\pi/2$
2	$\pi/2$		6	$\pi/2$		10	$\pi/2$		14	$\pi/2$	
3	π		7	π		11	π		15	π	
4	$3\pi/2$		8	$3\pi/2$		12	$3\pi/2$		16	$3\pi/2$	

For the values indicated in Table 2 for the angular variables Y_2 and Y_3 the variables X_2 and X_3 are constants and, consequently, the orbit's eccentricity e and inclination i are constants. The dependency between e and i is found from the fourth equation in (3.1), which, allowing for relations (2.1), we write as

$$\Phi(e, i) = \frac{\varkappa}{e\sin i\sqrt{a^*(1-e^2)}}\left[e\cos i\frac{\partial F_1^*}{\partial i} - (1-e^2)\sin i\frac{\partial F_1^*}{\partial e}\right] = 0$$

Apparently, this equation was first obtained by Charlier in connection with the three-body problem /7/ and was studied by many authors /8—13/ by numerical methods. In the problem being analyzed the solution of the third equation in (3.1) was determined numerically for eccentricity in the range $0 < e < 1.0$ and inclination in the range $0 < i < \pi$, where terms up to e^{10} were retained in the expansions of $C_k^{n,m}$ and $S_k^{n,m}$ (the theoretical limit of the convergence of the expansions mentioned equals $e = 0.6627$ /7/).

The form of the triaxial ellipsoid was characterized by a parameter $\varkappa$ defining the polar contraction of the ellipsoid and a parameter $\varkappa_1$ defining the equatorial contraction of the ellipsoid. Parameter $\varkappa_1$ was chosen in the form $\varkappa_1 \cong \varkappa^2$, i.e., the ratio between the contractions was retained the same as obtains for a planet of the Earth group. The calculation results are shown in Figs. 1—4 as the curves $\Phi(e, i) = 0$. In Fig.1 the curves correspond to commensurability 1/1 and to $\varkappa = 10^{-3}$. The solid curve refers to solutions of type 2, 4, 5, 7, 10, 12, 13, 15, while the dash-dotted curve refers to solutions of type 1, 3, 6, 8, 9, 11, 14, 16 (see Table 2). The curves in Figs.2 and 3 correspond to commensurability 1/1 and to $\varkappa = 10^{-2}$. The solid curves refer to solutions of type 1, 3, 9, 11 (in Fig.2) and type 2, 4, 10, 12 (in Fig.3), while the dash-dotted curves refer to solutions of type 6, 8, 14, 16 (in Fig.2) and type 5, 7, 13, 15 (in Fig.3). Finally, in Fig.4, corresponding to commensurability 1/4 and to $\varkappa = 10^{-2}$, the solid curves refer to solutions of type 1, 3, 9, 11, while the dash-dotted curves refer to solutions of type 6, 8, 14, 16; the curves corresponding to the solution of the equation $\partial F_1^* / \partial H = 0$ are shown by the dashed lines. Solutions of the equation $\Phi(e, i) = 0$ for other commensurabilities do not differ qualitatively from those shown in Figs.1—4. The curves shown in Figs.1—4 define a family of solutions for which $Y_2 = \text{const}$.

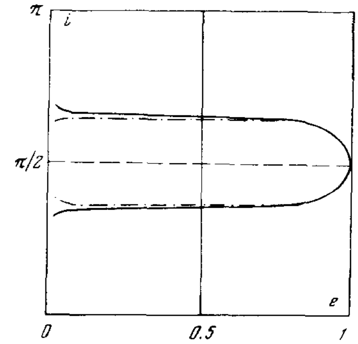

Fig.1

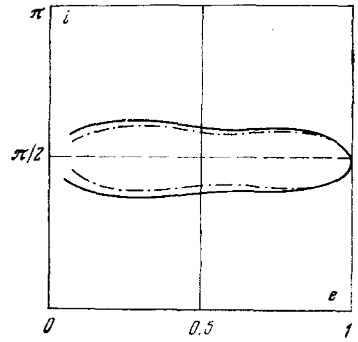

Fig.2

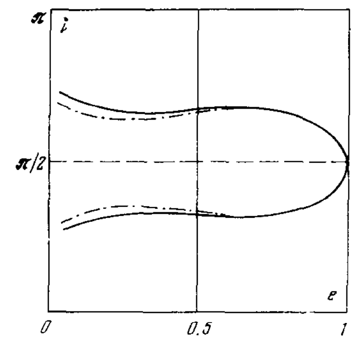

Fig.3

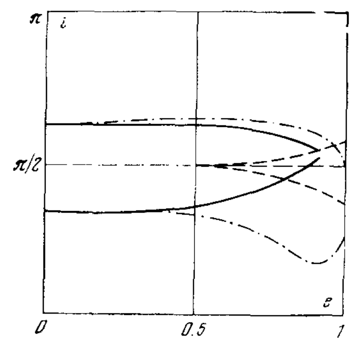

Fig.4

Let us now consider the sixth equation in (3.1). To obtain the steady-state solution $Y_3 = \text{const}$ of this equation it is necessary that the average motion n^* equal

$$n^* = s / p + \varkappa (\partial F_1^* / \partial L + (sp^{-1}) \partial F_1^* / \partial H) \tag{3.2}$$

In the expression we have taken into account that $\partial F_1^* / \partial G = 0$ on the curves $\Phi_1(e, i) = 0$. Finally, the variation of Y_1, being the fast variable, on a steady-state solution is determined by the second equation in (3.1) with a substitution of the expression found for n^*. As a result we obtain

$$Y_1^{\cdot} = (1 / p)(1 + \varkappa \partial F_1^* / \partial H)$$

The steady-state solutions obtained are synodic, i.e., after the passing of a period T the relative position of the material point and the triaxial ellipsoid in the rotating coordinate system is repeated. Although the arguments of the pericenter Y_2 and of the Delaunay anomaly Y_3 for the steady-state solution found retain constant values, the orbit's node is shifted even if we do not take into account the short-period terms of the perturbing function. Thus, the steady-state solutions found generate conditionally-periodic solutions. Analogous solutions in the three-body problem have come to be called solutions of Schwarzschild type /8/. These solutions differ from the Poincaré solutions of the third kind in which after the passing of one period the relative positions are repeated in the absolute coordinate system and, consequently, the orbit's node is fixed.

4. Conditionally-periodic and periodic solutions. Thus, with due regard to the short-period terms of the perturbing function F_1 the solutions are steady-state: the variation of the orbital elements are of an oscillatory nature relative to their steady-state values. The orbit's node precesses in the rotating coordinate system and, therefore, the solutions obtained are essentially conditionally-periodic. In order that the solutions found be periodic Poincaré solutions of the third kind, it is necessary that the condition $\Omega^{\cdot} = -\varkappa \partial F_1^* / \partial H = 0$ be fulfilled, i.e.,

$$\Phi_1(e, i) = [a^* (1 - e^2)]^{-1/2} (\sin i)^{-1} \partial F_1^* / \partial i = 0 \tag{4.1}$$

Equation (4.1) was solved numerically; the resulting curves $\Phi(e, i) = 0$ are shown by dashed lines in Figs. 1—4. Thus, for the parameter values e_0 and i_0, corresponding to points of intersection of curves $\Phi(e, i) = 0$ and $\Phi_1(e, i) = 0$, there will hold $Y_1^{\cdot} = 1/p$ and, after $(p+q) = s$ orbits of the mass point, the following will take their initial values: the variable Y_1, being a basic fast variable, and, consequently, the short-period perturbations in all the orbital elements corresponding to the steady-state solutions. The periodic solutions found exist for sufficiently small $\varkappa$. The period T_1 of the periodic solutions found equals sT, where T is the circulation period of the mass point for a given commensurability s/p. We remark that the period of the generating periodic solutions differs from period T since the average motion n^* differs from the exact commensurable value s/p on the generating (steady-state) solutions when $\partial F_1^* / \partial G = 0$ and $\partial F_1^* / \partial H = 0$.

5. Concluding notes.

Thus, in all the low-order commensurabilities examined there exist families of Schwarzschild-type periodic solutions, i.e., conditionally-periodic solutions. For parameter value $\varkappa = 10^{-3}$ (typical for the Earth-group planets) the majority of these solutions have an orbit inclination close to 1.108 and 2.033 (the precise value depends on e and $\varkappa$ and on the commensurability type s/p). The Poincaré periodic solutions of the third kind exist only for values e_0 and i_0 corresponding to the points of intersection of curves $\Phi(e, i) = 0$ (solid and dash-dotted lines) and curves $\Phi_1(e, i) = 0$ (dashed lines) in Fig.1—4. In view of the fact that to the points of intersection of the curves mentioned correspond the values $e_0 > 0.8$, exceeding the theoretical limit of convergence ($e = 0.6627$) of the expansions used, the question of the existence of periodic Poincaré solutions of the third kind in the strict sense remains open. In addition, the cases of small eccentricity and inclination, as well as the limit cases $e \to 0, e \to 1, i \to 0$ and $i \to \pi$, require a separate analysis.

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